MMP Learning Seminar.

Week 36.

Topics:

- Diophantine approximation.
- Boundedness of extremal lengths.
- Effective bpf Theorem.
- MMP with scaling.
- Shokurov's polytopes.

3.7. Diophantine approximations. $\pi : X \longrightarrow \operatorname{spec}(\mathbb{K})$.

Notation: $\pi : X \longrightarrow U$ projective morphism of normal quasi-projective varieties.

holds over U: morphism to U making all diagrams in the picture commute. $\overline{\mathbb{K}} = \mathbb{K}$, $\operatorname{char} \mathbb{K} = 0$.

Lemma 3.7.5. $(X, \Delta = A + B)$ log canonical, $A \geq 0$ and $B \geq 0$.

Assume $B_+(A/U)$ does not contain non-klt centers of (X, Δ). (X, Δ_0) is klt for some Δ_0. Then, we may find a klt pair $(X, \Delta' = A' + B')$, where $A' \geq 0$ general ample $\mathbb{Q}$-div over U, $B' \geq 0$, $K_X + \Delta' \sim_{\mathbb{R}, U} K_X + \Delta$.

Def:
$$B_+(L) = \bigcap_{m \in \mathbb{N}} B_s(mL - A)$$
for some ample A.

$B_+(L) \subsetneq X$, then L is big.

Idea: $A = A_0 + E_0$, E_0 does not contain non-klt (X, Δ)
$A' = A_0$ & $B' = B + E_0$.

Rational curves of low degree.

Theorem (Mori 1982): X smooth proj variety with $-K_X$ ample. Then, through any point $x \in X$, there is a rational curve C with:

$$0 \leq -K_X . C \leq \dim X + 1.$$

Theorem (Kawamata, 1991): $g: X \longrightarrow Y$ projective (X, Δ) log terminal & $-(K_X + \Delta)$ is g-ample. Then, every irreducible component of the $\mathrm{Ex}(g)$, is covered by a family of curves $\{C_\lambda\}_{\lambda \in \Lambda}$ so that $g(C_\lambda)$ are points and $-(K_X + \Delta) . C_\lambda \leq 2n$ for every $\lambda \in \Lambda$ and $n = \dim E$, the irreducible comp.

Lemma: H g-ample Cartier, $n = \dim E$ and $\nu: \bar{E} \longrightarrow E$ normalization, then

$$H^{n-1} \cdot (K_X + \Delta) \cdot E \geq (\nu^* H)^{n-1} \cdot K_{\bar{E}}.$$

Sketch of Kawamata's proof:

$Y_0 \subseteq Y$ generic linear section with codim $= \dim g(E)$

$X_0 = g^{-1}(Y_0)$, $g_0 : Y_0 \longrightarrow X_0$, H g_0-ample

Cartier, $\nu : \overline{E_0} \longrightarrow E_0$.

$C =$ int of $(n-1)$ general elements of $|\nu^* H|$

$M = -\nu^*(K_X + \Delta)$ ample Cartier on $\overline{E_0}$.

By Lemma, $\deg_C(K_{\overline{E_0}}|_C) < (-M \cdot C) < 0$.

If $n = 1$, since $R^1 g_{0*} \mathcal{O}_{X_0} = 0$, we have

$C = \overline{E_0} = E_0 \simeq \mathbb{P}^1$ and $-(K_X + \Delta) \cdot C < 2$.

Replace H with mH and assume C is not rational

KM Thm 1.3, for any $x \in C$, there exists a rational curve L on $\overline{E_0}$ passing through x s.t.

$$M \cdot L \leq 2n \frac{M \cdot C}{-K_{\overline{E_0}} \cdot C} < 2n.$$

$\square$

Question: R extremal $(K_X+\Delta)$-negative ray.

$$\ell(R) = \min \left\{ -(K_X+\Delta).C \;\middle|\; \begin{array}{l} C \text{ irreducible curve} \\ [C] \in R \end{array} \right\}$$

Boundedness of extremal lenghts. $\ell(R) \leq 2\dim X$.

Fix dimension n, describe the set of all possible $\ell(R)$.

$$(0, 2n]$$

3.8.1

Theorem: (X,Δ) log canonical, $K_X+\Delta$ $\mathbb{R}$-Cartier

Suppose $\exists \Delta_0$ s.t (X,Δ_0) is klt.

R a $(K_X+\Delta)$-neg extremal ray. then there exists

a rational curve Σ' spanning R, s.t

$$0 < -(K_X+\Delta).\Sigma' \leq 2n.$$

Here, $n = \dim X$.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[dl] \\ U & \end{tikzcd}$$

Sketch: i) Replace with klt (X,Δ)

ii) Assume $-(K_X+\Delta)$ is ample over the base

iii) approximate $\Delta \longleftarrow \Delta_i$ rational.

iv) produce Σ'_i with $-(K_X+\Delta_i).\Sigma_i \leq 2n$.

Sketch: i) Replace with klt (X, Δ)

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iii) approximate $\Delta \leftarrow \Delta_i$ rational.

iv) produce Σ'_i with $-(K_X + \Delta_i) . \Sigma_i \leq 2n$.

v) Pick A ample so that

$-(K_X + \Delta + A)$ is ample

Under these conditions, we will have $A . \Sigma'_i < 2n$

$$A \cdot \Sigma'_i = \underbrace{(K_X + \Delta_i + A) \cdot \Sigma'_i}_{<0} - \underbrace{(K_X + \Delta_i) \cdot \Sigma'_i}_{} < 2n$$

Assume $\Sigma' = \Sigma'_i$ stabilizes.

$$-(K_X + \Delta) . \Sigma' = \lim_i -(K_X + \Delta_i) . \Sigma'_i$$

$$= \lim_i -(K_X + \Delta_i) \cdot \Sigma' \leq 2n.$$

$\square$

Corollary 3.8.2. $(X, \Delta = A+B)$. lc

$A \geq 0$ ample $\mathbb{R}$-div over U, and $B \geq 0$

Assume (X, Δ_0) is klt for some $\Delta_0 \geq 0$.

Then, there are only finitely many $(K_X+\Delta)$-neg extremal rays $R_1, \ldots, R_k$.

Proof: R a $(K_X+\Delta)$-neg extremal ray.

$$-(K_X+B).R = -(K_X+\Delta).R + A.R > 0$$

Σ_i spanning R for which $-(K_X+B).\Sigma_i \leq 2n$

$$A.\Sigma_i = -(K_X+B).\Sigma_i + (K_X+\Delta).\Sigma_i \leq 2n.$$

Σ_i belongs to a bounded family.

$\pi: X \to U$.

Effective base point free theorem:

Theorem 3.9.1: Fix $n \in \mathbb{Z}_{>0}$. There exists $m>0$: $\boxed{D \text{ nef } \mathbb{R}\text{-divisor over } U}$, $aD-(K_X+\Delta)$ nef & big for some $a>0$ where (X,Δ) is klt & $\dim X = n$.

Then D is semiample over U. and if aD is Carbier, then maD is globally generated over U.

Proof: U affine, $D-(K_X+\Delta) \sim_{\mathbb{R},U} A+B$

replace $(X,\Delta) \rightsquigarrow (X,\Delta+\varepsilon B)$

$$\underbrace{D-(K_X+\Delta+\varepsilon B)}_{\text{ample}} = (1-\varepsilon)(\overbrace{D-(K_X+\Delta)}^{\text{big and nef}}) + \varepsilon(\underbrace{D-(K_X+\Delta+B)}_{\text{ample} \quad \sim_{\mathbb{R},U} A.})$$

Assume $D-(K_X+\Delta)$ ample

Replace $\Delta \rightsquigarrow \Delta'$ to assume is a $\mathbb{Q}$-divisors.

Pick A ample s.t. $D-(K_X+\Delta+A)$ ample,

replace (X,Δ) with $(X, \underbrace{\Delta+A}_{\text{big}})$

Assume Δ is big.

Apply 3.8.2. there are only finitely many $(K_X+\Delta)$-neg extremal rays $R_1, \dots, R_k$ of $\overline{NE}(X/U)$.

$$F = \{\alpha \in \overline{NE}(X) \mid D\cdot\alpha = 0\}$$

$\alpha \in F$, then $(K_X+\Delta)\cdot\alpha < 0$, α is spanned by some of the R_i's.

$V =$ smallest $\mathbb{Q}$-affine space of $\mathrm{WDiv}(X)$ containing D.

Then $\mathcal{C} = \{B \in V \mid B\cdot\alpha = 0,\ \forall \alpha \in F\}$ is rational.

This means that we can write

$$D = \sum r_p D_p \qquad r_p \text{ are areal numbers.}$$

so that each D_p is nef Cartier and $D_p - (K_X+\Delta)$ is ample

We conclude D_p are semiample + if aD_p is Cartier, then maD_p is bpf.

$\square$

Corollary: (X,Δ) klt, Δ big, then a min model is a good min model.

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Proof: $(X,\Delta) \dashrightarrow (X',\Delta')$ $\quad \Delta'$ big.

$K_{X'}+\Delta'$ is nef.

$$K_{X'}+\Delta' \sim_{\mathbb{R},U} K_{X'} + \underbrace{A'}_{\text{ample}} + \underset{\geq 0}{B'}$$

$$A' \sim_{\mathbb{R},U} \underbrace{\overbrace{(K_{X'}+\Delta')}^{\text{semiample}} - (K_{X'}+B')}_{\text{ample.}}$$

$K_{X'}+\Delta'$ is semiample so is a good minimal model $\square$.

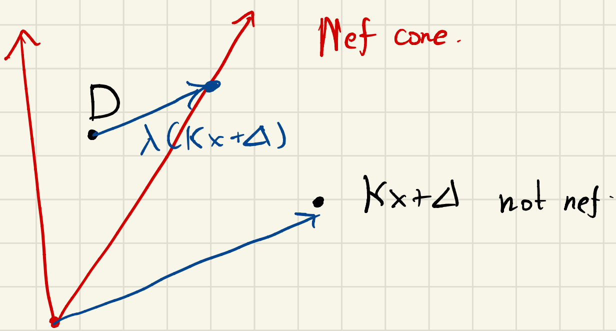
MMP with scaling:

Lemma 3.7.10: $(X, A+B)$ klt, $A \geq 0$ big & $B \geq 0$.

D nef Cartier over U, $K_X + \Delta$ is not nef over U. Set

$$\lambda = \sup\{\mu \mid D + \mu(K_X+\Delta) \text{ nef over } U\}$$

Then, there exists a $(K_X+\Delta)$-neg extremal ray R over U s.t. $(D + \lambda(K_X+\Delta)) \cdot R = 0$.



Proof: By (3.7.5), we may assume A ample.

Only finitely many $(K_X+\Delta)$-neg extremal rays $R_1, \dots, R_k$ over U.

Σ_i generate R_i.

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Only finitely many $(K_X+\Delta)$-neg extremal rays $R_1, \dots, R_k$ over U.

Σ_i generate R_i. Let

$$\boxed{\lambda = \mu} = \min_i \frac{D \cdot \Sigma_i}{-(K_X+\Delta)\cdot \Sigma_i}.$$

$D+\mu(K_X+\Delta)$ nef over U. Furthermore, the intersection with at least one of the Σ_i's is zero $\square$.

C is $(K_X+\Delta)$-non neg

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MMP with scaling: (Assume existence of flips).

C nef Cartier divisor.

$(X, \Delta + C = \boxed{S} + \widetilde{A} + B + C)$, dlt pair $\lfloor \Delta \rfloor = S$.

$A \geq 0$ big divisor s.t. $B_+(A/U)$ does not contain strata of S. $B \geq 0$.

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$(K_X + \Delta)$- neg extremal ray which is $(K_X + \Delta + \lambda C)$ - trivial

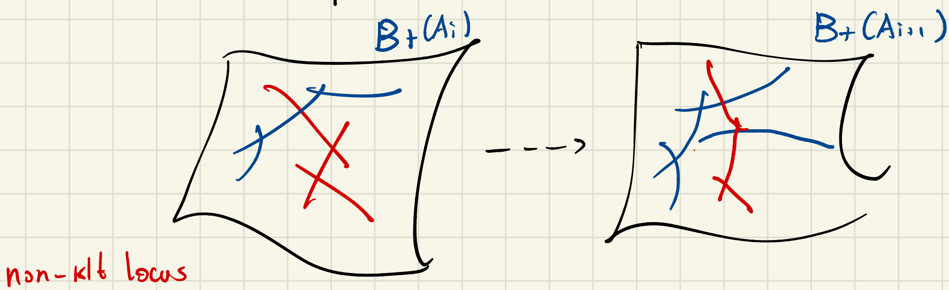
$$X \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} X_3 \overset{\phi_3}{\dashrightarrow} X_4 \overset{\phi_4}{\dashrightarrow} \cdots$$

$\Delta_i = (\phi_{i-1})_* \Delta_{i-1}$, and $C_i = (\phi_{i-1})_* C_{i-1}$.

$K_{X_i} + \Delta_i + \lambda_i C_i$ nef over U and we have a sequence of real numbers $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots$

Remark 1: It preserves the dlt condition.

Remark 2: It preserves the condition on $B_+(A_i/U)$



Shokurov's polytopes:

$\mathcal{L}_A(V)$, $\mathcal{N}_A(V)$

$\Delta \in \mathcal{L}_A(V)$

(X,Δ) lc + $K_X+\Delta$ nef.

$\Delta \geq A$

$\Delta = A+B$

log canonical

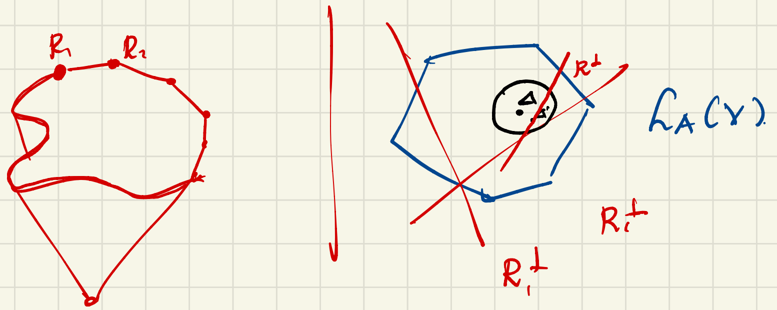
(X,Δ)

Shokurov's Polytopes:

$\pi : X \longrightarrow U.$

Thm 3.11.1: $V \subseteq WDiv_{\mathbb{R}}(X)$ finite dim over $\mathbb{Q}$. $\mathbb{Q}$-ample divisor A over U. (X, Δ_0) is klt for some Δ_0. The set of hyperplanes $R^\perp$ is finite in $\mathcal{L}_A(V)$ as R ranges over extremal rays of $\overline{NE}(X/U)$.

In particular, $\mathcal{N}_{A,\pi}(V)$ is a rational polytope.



Proof: $\mathcal{L}_A(V)$ compact. It suffices to prove locally around $\Delta \in \mathcal{L}_A(V)$.

Assume (X, Δ). Fix $\varepsilon > 0$ s.t. if $\Delta' \in \mathcal{L}_A(X)$ and $\|\Delta - \Delta'\| < \varepsilon$, then $\Delta' - \Delta + A/2$ is ample over U.

R is extremal with $(K_X + \Delta') \cdot \boxed{R} = 0$, $\Delta' \in \mathcal{L}_A(X)$, $\|\Delta - \Delta'\| < \varepsilon$. We have

$\Delta = A + B$

$\Delta = A/2 + B.$

$$\boxed{(K_X + \Delta - A/2) \cdot R} = (K_X + \Delta') \cdot R - (\Delta' - \Delta + A/2) \cdot R < 0 \quad \square.$$